A three-dimensional solution for waves in the lee of mountains

By G. D. CRAPPER
Department of Mathematics, University of Manchester

(Received 14 November 1958)

This paper presents a three-dimensional small-perturbation approach to the problem of waves produced in a statically stable stratified air stream flowing over a mountain. The fundamental solution for a doublet disturbance in an air stream in which the parameter $l^2 = g\beta/V^2$ is constant is calculated, and then is extended to that for a disturbance due to a circular mountain in the same air stream. A simple approximation to the known two-dimensional flow over an infinite ridge is also given. The second ('upper') boundary condition for the solutions is determined in a rigorous analytical manner, assuming the presence of small friction forces, or, alternatively, of time dependence. It is hoped that this will settle the controversy which exists over the choice of this condition.

The results show that the behaviour due to a doublet is peculiar and not truly representative of that due to a mountain. The latter shows waves which decay down-stream and are contained in a strip, the width of the strip being determined by the radius of the mountain. An interesting result is that the circular mountain can give rise to waves which have greater amplitude than those produced by the infinite ridge under the same conditions.

In some previous papers the waves produced by the infinite ridge have been neglected, but the present paper shows that in many cases this procedure is not justifiable. The detailed solution for the waves behind a circular mountain has a form which emphasizes the importance for lee-wave production of 'resonance' between the width of the mountain and the characteristic length l^{-1} of the air stream.

---

1. Introduction

The problem of airflow over mountains, and, in particular, the formation of stationary atmospheric lee waves, has received considerable theoretical attention in recent years. Many examples of actual wave systems have been described in various meteorological journals, observations being made from the ground, on the clouds which often form in the wave crests, and from gliders by finding the up-currents. In California, where the Sierra Nevada produces what is probably the most spectacular of all lee-wave systems, many glider flights to altitudes of 30,000 to 40,000 ft. have been made. Waves with amplitudes and velocity components large enough to affect powered aircraft are often encountered, and this alone is a sufficient reason for a thorough investigation. The results of observations are well summed up by Corby (1954), who also gives a survey of the theoretical treatments then available. These are essentially of two types, depending on the nature of the air stream considered. All are linear perturbation theories, and in the first place they are restricted to the two-dimensional flow over an infinitely long ridge.

The first investigations, restricted to an air stream in which both velocity and static stability are uniform, were by Lyra (1943) and Queney (1947). Lyra used a complicated Green's function method to work out the flow over a small ridge of rectangular cross-section, while Queney used a line disturbance on the ground, later (1948) extending this by use of Fourier Transforms to obtain the flow over the ridge given by

$$\zeta(x,0) = \frac{Hb^2}{(b^2+x^2)}, \tag{1}$$

where $\zeta(x,z)$ is the vertical displacement of the streamline from $z=$ const., z being height, with the ground at $z=0$, and x the horizontal (stream direction) co-ordinate. The height of the ridge is H, and b is the half-width of the ridge at height $\frac{1}{2}H$. The results of all these investigations are basically the same, giving waves behind the ridge, with amplitudes which increase with height and fall off down-stream. Queney does show, however, that if b is large enough the waves are so small as to be negligible. (Queney also considers motion on a much larger scale, in which the effects of the earth's rotation are included, but this is beyond the scope of the present paper.) The waves shown by these theories are unlike those which are usually observed in practice. These latter reach their maximum amplitude at some middle level, and their rate of decay with distance down-stream is slow, and can reasonably be attributed to friction, which is, of course, neglected in all these theories.

Scorer (1949) decided that the 'uniform air stream' assumption of Lyra and Queney was too unrealistic, and he considered an air stream consisting of two layers, in each of which the parameter

$$l^2 = \frac{g\beta}{V^2} - \frac{V''}{V} \tag{2}$$

is constant; here g is the gravitational acceleration, $V(z)$ the main-stream velocity, $\beta = \Theta'/\Theta$ is the parameter governing the static stability of the atmosphere (where $\Theta(z)$ is potential temperature), and primes denote differentiation with respect to z. Using the ridge shape given by (1), and again the Fourier Transform method, Scorer finds that waves of the type usually observed occur if

$$l_l^2 - l_u^2 > \frac{\pi^2}{4h^2}, \tag{3}$$

where l_l, l_u are the lower and upper layer values of l, and h is the depth of the lower layer. In other words, l^2 has to decrease upwards in a relatively short distance by a sufficiently large amount. Thus it would appear that the nature of the profile is the factor which determines the type of waves that appear. Mathematically, the effect of the discontinuity in l^2 is to introduce a pole singularity into the Fourier Integral for the displacement of the streamlines, and it is this pole which gives the non-decaying waves. In Queney's theory for the uniform air stream the only singularity is a branch point.

Only Scorer & Wilkinson (1956) have done any work on mountain waves in three dimensions, integrating the two-layer solution over an infinite number of ridges at all angles to the undisturbed stream to produce the flow over circular and oval hills. This time the waves are smaller and are contained in a wedge behind the hill. They are very similar to water waves behind a ship.

In another paper, Scorer (1953) has considered the flow over a ridge when l^2 is constant (which includes Lyra's and Queney's solutions), but he has neglected the wave term on the grounds that the half-width b of the ridge is large enough for it to be negligibly small. Scorer then goes on in the same paper to a further discussion of lee waves, this time with a three-layer model, but he does not point out that these waves would also be negligible (as will appear below) for the values of b for which the first part of the paper is valid. Corby & Wallington (1956) have investigated the variations of the amplitude of the two-layer waves, and they show that if either b is increased with H constant, so that the slope of the surface is decreased, or if b and H are increased in the same proportion, so that the slope remains constant, the amplitude reaches a sharp maximum and then falls off rapidly. If the slope is allowed to decrease, the maximum occurs for smaller b than that needed if the slope is kept constant, although the rates of fall-off from the maxima are similar. Thus the width is the important parameter of the ridge as far as wave amplitude is concerned, although increasing the height of the ridge for fixed b increases the amplitude in proportion. Corby & Wallington have compared these sharp maxima with a 'resonance' of the air stream with the mountain. The present paper will show that, in both two and three dimensions, the one-layer flow with l^2 constant gives this same effect, and in fact, in two-dimensions, the one- and two-layer solutions give exactly the same curves of amplitude against mountain width.

Scorer's neglect of waves in the first part of his paper, where, to be sure, they may be difficult to justify by observation, but not in the second part, where they are well supported by observation, is consequently misleading. Also, on general grounds, one would expect waves to be formed when any stable air stream is disturbed, and the l^2-profile must therefore determine the kind of waves and not whether or not they exist.

Different writers have disagreed in the choice of the so-called upper boundary condition. The wave equation (31) below requires two boundary conditions, the first being automatically determined by the choice of the ground shape, and the second, which is arbitrary, being needed to remove any wave train which may otherwise occur up-stream. The present paper uses two methods which are essentially the same, as will be seen in § 2. The first device, originally used by Rayleigh, assumes a small amount of friction (see Lamb 1932, § 242), which is eventually allowed to tend to zero, and the second assumes a time dependence, finally letting time tend to infinity to give the steady-state solution. Each of these methods gives the result used by Queney. Corby & Sawyer (1958*a*), who assume the atmosphere has a rigid lid on top and then let the height of the lid become infinite, reach the same conclusion. Scorer, in all his papers, considers energy flow, and chooses the other alternative boundary condition—a change of sign only. The controversy over this condition has gone on for some time (see, for instance, Scorer 1954, 1958; Corby & Sawyer 1958*a*, *b*), but it may perhaps be hoped that the rigorous discussion in this paper will be accepted as establishing the former condition beyond any doubt.

It seemed to the author that the existing mathematical theories of lee waves, especially in three dimensions, were not altogether satisfactory, and the present work was undertaken to re-examine the whole question. We begin, following the method used in the absence of gravitational effects by Lighthill (1957), by determining the three-dimensional flow due to a weak source in the given air stream $V(z)$. This 'fundamental solution' has proved to have considerable value in several other problems, and it therefore seems worth while giving it here, even though the results of § 5 show that it is not truly representative of the behaviour produced by a mountain in the same air stream. After taking Fourier transforms of the equations of motion, a single equation for the disturbance is derived. The work is then restricted to the case when $l^2 =$ const. and $V'' = 0$ throughout the atmosphere, in order to solve the equation reasonably simply.

From this solution we infer the solution for a doublet in the stream or x-direction in the form of a double Fourier integral for the streamline disturbance, the second boundary condition being found as stated above. The first integration can be performed exactly, but the second is much more difficult and has to be approached by suitably deforming the path of integration. It gives a non-wavy disturbance, for which an asymptotic expansion, symmetrical up-stream and down-stream of the doublet, is found for large x, and also wave terms which appear only down-stream. Three separate approximations to the latter have been found, all for large x, one for y small (where y is the horizontal co-ordinate perpendicular to the stream), one for larger y, and one for large z. The mathematics involved in finding the one for small y (actually given second in the paper) is somewhat complicated, and is given in an appendix, the results simply being quoted in § 3.

The behaviour of the wave term for the doublet disturbance is rather peculiar. On $y = 0$, immediately behind the doublet, the waves decay down-stream like $x^{-\frac{1}{2}}$, but as y increases the rate of decay decreases, and the dominant term in the larger y approximation has no decay at all. As height is increased, the amplitude of the waves tends to a finite value; if $V'(z) > 0$ this may be zero. The wave crests in planes at constant height are hyperbolic in shape, their asymptotes all passing through the origin and making a smaller angle with the x-axis the further the waves are down-stream. The changing behaviour as y increases is impossible to explain physically, and it is assumed to be due to the unrealistic character of the doublet which, being a point disturbance, contains no significant length. Accordingly, the solution for larger y is extended to the case of a horizontal line of doublets perpendicular to the stream, which has such a length, and this shows no unexpected behaviour, the waves for y large compared with half the length of the line decaying simply as x^{-1}.

Approximations to Queney's solution for the infinite line doublet and the ridge are given in § 4 to enable comparisons to be made, and in § 5 the three-dimensional theory is extended to cover, as far as possible, the circular hill given by

$$\zeta(x, y, 0) = \frac{Ha^3}{(a^2+x^2+y^2)^{\frac{3}{2}}}. \tag{4}$$

An infinite line of these hills gives the ridge (1). The integral is more difficult to deal with in this case, even though there are less singularities which produce waves. A Taylor-series approach gives a solution valid for large x, but when z is large the method becomes useless and the behaviour at large heights remains unknown. The non-wave disturbance is no longer symmetrical, and now the waves decay like $x^{-\frac{1}{2}}$, as do those given by the ridge; the amplitude increases with height as far as the solution holds. As y increases, the amplitude falls off very rapidly, the actual rate of fall-off being largely determined by a. It seems that the extra singularities of the doublet integral are responsible for its peculiar behaviour, and they have been removed by bringing the length a into the mountain solution.

The waves are not contained in a wedge, as might have been expected, but in a strip. Thus the energy of the waves is concentrated near the plane $y = 0$, immediately behind the mountain, and this probably accounts for the fact that waves can occur in this plane with amplitude greater than that of the waves given by the infinite ridge.

As pointed out above, the circular mountain solution gives the resonance effect with change of mountain radius. The maxima for waves in the plane $y = 0$ are rather more sharp than those shown by the infinite ridge, although the peaks become more rounded as y is increased The optimum value of a depends on the value of y chosen, but this effect is unimportant because of the rapid fall-off of amplitude.

We can conclude, then, that any mountain will produce lee waves in a stably stratified air stream, the form of the waves depending on the shape of the mountain and the shape of the l^2-profile. The amplitudes will in many cases be negligible, but this is determined solely by the width of the mountain in relation to the characteristic wavelength of the air stream, and is otherwise independent of the actual shape of the l^2-profile.

2. The equations of motion

We shall assume initially that we have a weak source, of strength m, situated at the origin, and we shall consider the small (linear) disturbances which it produces in a steady main-stream flowing horizontally in the x-direction. We shall write

$$\left.\begin{aligned} \text{velocity} &= (V(z)+u, v, w), \\ \text{density} &= \rho(z)+s, \\ \text{entropy} &= S(z)+\phi, \\ \text{pressure} &= p(z)+\theta, \end{aligned}\right\} \tag{5}$$

where the undisturbed main-stream values $V(z)$, $\rho(z)$, $S(z)$, $p(z)$ are functions of height z only, and the perturbation terms u, v, w, s, ϕ, θ are all functions of the rectangular Cartesian co-ordinates x, y, z. The perturbation terms are assumed to be zero at large distances up-stream, and squares and products of them will be neglected everywhere.

Far up-stream there is no flow in the z-direction, so that

$$p'(z)+g\rho(z) = 0, \tag{6}$$

where g is the gravitational acceleration. Throughout this section primes denote differentiation with respect to z. The equation of state gives us the two equations

$$p = \kappa\rho^{\gamma}\, e^{S/C_V}, \tag{7}$$

and

$$\theta = \kappa\rho^{\gamma}e^{S/C_V}\left(\frac{\gamma s}{\rho}+\frac{\phi}{C_V}\right) \tag{8}$$

for the undisturbed and disturbed regions respectively. Here κ is a constant, C_V is the specific heat of air at constant volume, and γ is the ratio of the specific heats of air. C_V and γ are assumed to be constant.

The linearized momentum equations are

$$\mu\rho u+\rho V u_x+\rho V' w+\theta_x = 0, \tag{9}$$

$$\mu\rho v+\rho V v_x+\theta_y = 0, \tag{10}$$

$$\mu\rho w+\rho V w_x+gs+\theta' = 0, \tag{11}$$

where suffixes denote derivatives. The terms $\mu\rho u$, $\mu\rho v$, $\mu\rho w$ are included to give us a rigorous analytical method of eliminating any disturbance which may otherwise appear far up-stream. They can be considered as friction terms, and as such they were first introduced by Rayleigh (Lamb 1932, § 242), or as the operational representations of the terms $\rho\,\partial u/\partial t$, $\rho\,\partial v/\partial t$, $\rho\,\partial w/\partial t$, in which case μ is the Heaviside operator. We will eventually let $\mu\to 0$; in the second case this corresponds to time $t\to\infty$, giving the ultimate steady flow solution.

We have two further equations, namely

$$\frac{D}{Dt}(S+\phi) = 0, \tag{12}$$

which reduces to

$$\mu\phi+V\phi_x+wS' = 0, \tag{13}$$

and the equation of continuity, which with a source of strength m at the origin is

$$\frac{\mu s}{\rho}+\frac{V s_x}{\rho}+\frac{w\rho'}{\rho}+u_x+v_y+w' = m\delta(x)\,\delta(y)\,\delta(z). \tag{14}$$

Here $\delta(\;)$ is the Dirac delta function. Terms similar to that on the right of (14) do not appear in (9), (10), (11) as the source is considered to be a source of mass but not of momentum. The μ terms in (13) and (14) appear only if μ is taken to be the Heaviside operator, but we shall see below that omitting them does not affect our result.

We shall now take Fourier transforms, of the form

$$w(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\xi x+\eta y)\}\,w_0(z,\xi,\eta)\,d\xi\,d\eta, \tag{15}$$

of equations (8), (9), (10), (11), (13) and (14), and then combine them to find an equation for w_0. Suffix zero will always denote the transform. Equations (9) and (10) become

$$\rho(\mu+Vi\xi)\,u_0+\rho V' w_0+i\xi\theta_0 = 0, \tag{16}$$

$$\rho(\mu+Vi\xi)\,v_0+i\eta\theta_0 = 0. \tag{17}$$

If we multiply (16) by $i\xi$ and (17) by $i\eta$ and then add them together, we can remove the terms in u_0 and v_0 by substituting from

$$(\mu+Vi\xi)\frac{s_0}{\rho}+\frac{w_0\rho'}{\rho}+u_0 i\xi+v_0 i\eta+w_0' = \frac{m}{4\pi^2}\delta(z), \tag{18}$$

the transform of (14). Writing

$$k^2 = \xi^2+\eta^2, \tag{19}$$

this gives

$$\rho(\mu+Vi\xi)\left\{\frac{m}{4\pi^2}\delta(z)-w_0'-\frac{w_0\rho'}{\rho}\right\}+i\xi\rho V'w_0 = k^2\theta_0+(\mu+Vi\xi)^2 s_0. \tag{20}$$

From the transforms of (8) and (13), we get

$$s_0 = \frac{\theta_0}{c^2}+\frac{\beta\rho w_0}{(\mu+Vi\xi)}, \tag{21}$$

where

$$c = (\gamma\kappa\rho^{\gamma-1}e^{S/C_v})^{\frac{1}{2}} \tag{22}$$

is the velocity of sound, and

$$\beta = \frac{S'}{C_V\gamma} = \frac{\Theta'}{\Theta}, \tag{23}$$

Θ being potential temperature (β is the parameter which defines the static stability of the atmosphere). Using (21), the right-hand side of (20) becomes

$$\theta_0\left\{k^2+\frac{(\mu+Vi\xi)^2}{c^2}\right\}+(\mu+Vi\xi)\beta\rho w_0, \tag{24}$$

and we can neglect the second term in the curled bracket because $k^2 = O(\xi^2)$ and $(\mu+Vi\xi)^2/c^2 = O(\xi^2V^2/c^2) \ll O(\xi^2)$, as $V \ll c$ in the atmosphere. We now differentiate (20) (with the right-hand side (24)) with respect to z, and remove the θ_0' term by using the transform of (11), (20) and (24), and (21). Then, dividing by $\rho(\mu+Vi\xi)$, using the theorems

$$\left.\begin{aligned} f(z)\,\delta(z) &= f(0)\,\delta(z),\\ f(z)\,\delta'(z) &= f(0)\,\delta'(z)-f'(0)\,\delta(z),\end{aligned}\right\} \tag{25}$$

and using the fact that

$$\frac{g}{c^2}+\frac{\rho'}{\rho}+\beta = 0, \tag{26}$$

we have

$$w_0''-w_0'\left\{\frac{g}{c^2}+\beta\right\}+w_0\left\{-k^2-\frac{k^2g\beta}{(\mu+Vi\xi)^2}+\frac{g\beta}{c^2}-\frac{g\rho'}{c^2\rho}-\frac{\gamma g^2}{c^4}-\frac{Vi\xi}{(\mu+Vi\xi)}\left[\frac{gV'}{c^2V}+\frac{V''}{V}-\frac{\beta V'}{V}\right]\right\}$$
$$= \frac{m}{4\pi^2}\left\{\delta'(z)+\left[\frac{V'(0)\,i\xi}{(\mu+V(0)\,i\xi)}-\beta(0)\right]\delta(z)\right\}. \tag{27}$$

The right-hand side of (27) is zero for $z > 0$ and $z < 0$, but at the origin we have a discontinuity such that

$$\left.\begin{aligned} w_0(+0)-w_0(-0) &= \frac{m}{4\pi^2}\\ \text{and}\quad w_0'(+0)-w_0'(-0) &= \frac{m}{4\pi^2}\left[\frac{V'(0)\,i\xi}{(\mu+V(0)\,i\xi)}-\beta(0)\right].\end{aligned}\right\} \tag{28}$$

If we had omitted the μ-terms in (13) and (14), the only difference in (27) would be that the $(\mu+Vi\xi)^2$ would be replaced by $(\mu+Vi\xi)\,Vi\xi$.

We can simplify (27) considerably by neglecting further terms. In the atmosphere $VV' \ll g$, $V^2 \ll c^2$ and g/c^2 and ρ'/ρ are $O(\beta)$, so that

$$\frac{g\beta}{V^2} \gg \frac{g\beta}{c^2}, \quad \frac{g\rho'}{c^2\rho}, \quad \frac{\gamma g^2}{c^4}, \quad \frac{gV'}{c^2 V}, \quad \frac{\beta V'}{V}, \tag{29}$$

and our equation becomes, for $z \neq 0$,

$$w_0'' - w_0'\left\{\frac{g}{c^2} + \beta\right\} + w_0\left\{-k^2 - \frac{k^2 g\beta}{(\mu + Vi\xi)^2} - \frac{V'' i\xi}{(\mu + Vi\xi)}\right\} = 0, \tag{30}$$

with the same discontinuities at the origin.

It is convenient at this stage to put $\mu = 0$ and to consider the effect of non-zero μ later. We then have

$$w_0'' - w_0'\left\{\frac{g}{c^2} + \beta\right\} + w_0\left\{\frac{k^2 g\beta}{\xi^2 V^2} - k^2 - \frac{V''}{V}\right\} = 0. \tag{31}$$

3. The solution for a doublet disturbance

Lee waves are usually found to produce a down-flow on lee slopes, delaying separation from the surface (Scorer 1955) and, for the purposes of this paper, this should make a doublet better than a source as a representation of a mountain. We will therefore go on to find the flow due to a doublet, as we can derive the basic equation very simply from (30).

The vertical perturbation velocity given by a doublet in the x-direction at the origin is simply the derivative with respect to x of that given by the source. Moreover, the vertical displacement of a streamline from its original position is

$$\zeta(x, y, z) = \int \frac{w}{V} dx, \tag{32}$$

and so its Fourier transform ζ_0 is another solution of (30). It follows that simply writing $w_0/V = \zeta_0$ will give the streamline disturbance produced by the doublet, since then the differentiation and integration with respect to x cancel.

In order to solve (31) (i.e. (30) with $\mu = 0$) reasonably simply we now assume that we have a simple-shear main stream, so that

$$V''(z) = 0, \tag{33}$$

and we assume that

$$l^2 = g\beta/V^2 = \text{const.} \tag{34}$$

throughout the whole depth of the atmosphere. These are considerable restrictions, but they have had to be retained for the rest of the paper. Both V'' and l^2 are certain to vary in the atmosphere, but even so the results should show, in part at least, how the presence of stable stratification affects the flow.

The solution of (31) can now be written

$$\zeta_0 = Ae^{\lambda_1 z} + Be^{\lambda_2 z}, \tag{35}$$

in which A and B are arbitrary constants, and

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\left(\frac{g}{c^2} + \beta\right) \pm \sqrt{\left\{\left(\frac{g}{c^2} + \beta\right)^2 + 4K^2\right\}}\right], \tag{36}$$

where

$$K = k\sqrt{\left(1 - \frac{l^2}{\xi^2}\right)}. \tag{37}$$

We choose the square root in (37) so that the real part $\mathscr{R}(K)$ is positive. As $K^2 = O(g\beta/V^2)$, we can in (36) neglect the $(g/c^2+\beta)^2$ inside the square root. Now we must have $\zeta_0 \to 0$ as $z \to \infty$, where possible, and also $\zeta_0(-0) = 0$ on the ground (the horizontal plane $z = 0$). Thus, for K real, we must have

$$\zeta_0 = \frac{m}{4\pi^2 V} \exp\{\tfrac{1}{2}(g/c^2+\beta)z - Kz\}. \tag{38}$$

The effect of the term $\exp\{\frac{1}{2}(g/c^2+\beta)z\}$ is small compared with that of e^{-Kz}, and Scorer omits it, whilst Queney approximates it by $[\rho(0)/\rho(z)]^{\frac{1}{2}}$. Here we will leave it out in the working and consider its effect later. We will necessarily have to take (38) as the solution for all values of K, and the positions of the singularities in the ξ Fourier Integral will determine the solution for imaginary K. We then have

$$\zeta = \frac{m}{4\pi^2 V}\int_{-\infty}^{\infty} e^{i\xi x}\int_{-\infty}^{\infty} \exp\{i\eta y - z\{(\xi^2-l^2)(1+\eta^2/\xi^2)\}^{\frac{1}{2}}\}\, d\eta\, d\xi. \tag{39}$$

The integration with respect to η can be carried out explicitly, although it has to be done in two parts.

First, we must consider the range $|\xi| > zl/t$, where

$$t = \sqrt{(y^2+z^2)}. \tag{40}$$

Substituting

$$\eta = \xi \sinh \tau, \tag{41}$$

the η integral in (39) becomes

$$\int_{-\infty}^{\infty} \exp\{i\xi y \sinh\tau - z\sqrt{(\xi^2-l^2)}\cosh\tau\}\,\xi\,(\operatorname{sgn}\xi)\cosh\tau\, d\tau. \tag{42}$$

The function sgn ξ, equal to $+1$ when $\mathscr{R}(\xi) > 0$ and to -1 when $\mathscr{R}(\xi) < 0$ (at the moment we are only concerned with ξ real) is necessary because, otherwise, if $\xi < 0$ the limits of the integral would be interchanged. We can write (42) as

$$-\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-l^2)}}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \exp\{i\xi y\sinh\tau - z\sqrt{(\xi^2-l^2)}\cosh\tau\}\,d\tau, \tag{43}$$

$$= -\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-l^2)}}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \exp\{ir_1 \sinh(\tau+i\psi_1)\}\,d\tau, \tag{44}$$

where

$$r_1 = +\sqrt{\{\xi^2y^2 + z^2(\xi^2-l^2)\}}, \tag{45}$$

and

$$\tan\psi_1 = \frac{z\sqrt{(\xi^2-l^2)}}{\xi y}. \tag{46}$$

We have previously defined $\mathscr{R}(K) > 0$ in (37), making $\sqrt{(\xi^2-l^2)} > 0$ when $|\xi| > l$, and now we define $r_1 > 0$ so that ψ_1 is real and $\sin\psi_1 > 0$ ($|\xi| > l$), or ψ_1 is pure imaginary ($zl/t < |\xi| < l$). With these values of ψ_1, and putting $\tau + i\psi_1 = p_1$, (44) is equal to

$$-\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-l^2)}}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \exp(-ir_1\sinh p_1)\,dp_1, \tag{47}$$

$$= -\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-l^2)}}\frac{\partial}{\partial z}\{2K_0(r_1)\}, \tag{48}$$

since r_1 is real and positive (Watson 1944, § 6.22, equation (10)). (Here K_0 is the Bessel function of imaginary argument.) Thus for $|\xi| > zl/t$, (42) is

$$\frac{2\xi\,(\operatorname{sgn}\xi)\,z\sqrt{(\xi^2-l^2)}\,K_1\{\sqrt{(\xi^2t^2-z^2l^2)}\}}{\sqrt{(\xi^2t^2-z^2l^2)}}. \tag{49}$$

This function has singularities at $\xi = \pm l$ and $\xi = \pm zl/t$, and before we can consider the η-integral for $|\xi| < zl/t$ (for which r_1 is pure imaginary), we must determine the path of the ξ-integration as it passes these singularities. If instead of putting $\mu = 0$ in (30), we put $\mu = V\epsilon$ (which is justifiable since we shall let $\mu \to 0$, i.e. $\epsilon \to 0$), this is equivalent to replacing our l^2 by $l^2\xi^2/(\xi - i\epsilon)^2$. The singularities then occur where

$$\xi^2t^2 - \frac{z^2l^2\xi^2}{(\xi-i\epsilon)^2} = 0 \tag{50}$$

and

$$\xi^2 - \frac{l^2\xi^2}{(\xi-i\epsilon)^2} = 0, \tag{51}$$

i.e. at $\xi = 0$, $\xi = i\epsilon \pm zl/t$, $\xi = i\epsilon \pm l$ and $\xi = i\epsilon$. Thus the singularities, apart from the one at $\xi = 0$, are in the upper half of the ξ-plane, and therefore when we put $\mu = 0$ we must take the path of the ξ-integral on the lower side of the real axis. It follows that the singularities other than that at $\xi = 0$ do not contribute to the ξ-integral for $x < 0$. Treating the μ-terms as friction forces, and hence omitting them in (13) and (14), leads to the replacing of l^2 by $l^2\xi/(\xi - i\epsilon)$, with the same result. With this path of integration, then, we have

$$\sqrt{(\xi^2-l^2)} = \begin{cases} -i\sqrt{(l^2-\xi^2)} & 0 < \xi < l, \\ +i\sqrt{(l^2-\xi^2)} & -l < \xi < 0 \end{cases} \tag{52}$$

and

$$r_1 = \begin{cases} -ir_2 & 0 < \xi < zl/t, \\ +ir_2 & -zl/t < \xi < 0, \end{cases} \tag{53}$$

where

$$r_2 = +\sqrt{(z^2l^2-\xi^2t^2)}. \tag{54}$$

Thus our boundary condition of friction or of time dependence has determined the position of the singularities by a rigorous analytical method, and has given the same result as the upper boundary condition used by Queney (1947) and other writers. Scorer's condition, which he uses in all his papers, is given by taking the complex conjugate of (52) ((53) does not occur in two-dimensional solutions), and therefore takes the path of integration above the singularities. As we shall see, these singularities account for the presence of waves, and it follows that Scorer's condition should result in waves appearing up-stream but not down-stream. It is consequently difficult to see how Scorer's condition can possibly be correct.

We can now return to the η-integral. For $0 < \xi < zl/t$, (43) is

$$-\frac{i\xi}{\sqrt{(l^2-\xi^2)}}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\exp\{i\xi y\sinh\tau + iz\sqrt{(l^2-\xi^2)}\cosh\tau\}\,d\tau, \tag{55}$$

$$= -\frac{i\xi}{\sqrt{(l^2-\xi^2)}}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\exp(ir_2\cosh p_2)\,dp_2, \tag{56}$$

with

$$\tau + \psi_2 = p_2 \tag{57}$$

and

$$\tanh\psi_2 = \frac{\xi y}{z\sqrt{(l^2-\xi^2)}} \quad (\psi_2\ \text{real}). \tag{58}$$

According to Watson (1944, § 6.21, equation (10)), (56) is

$$-\frac{i\xi}{\sqrt{(l^2-\xi^2)}}\frac{\partial}{\partial z}\{\pi i H_0^{(1)}(r_2)\}, \tag{59}$$

$$= -\frac{i\xi}{\sqrt{(l^2-\xi^2)}}\frac{\partial}{\partial z}\{2K_0(-ir_2)\}. \tag{60}$$

The range $-zl/t < \xi < 0$ gives the complex conjugate of (55), but as the relation between r_1 and r_2 also changes, we have the result (49) for all ξ, and

$$\zeta = \frac{m}{2\pi^2 V}\int_{-\infty}^{\infty} e^{i\xi x}\frac{\xi\,(\operatorname{sgn}\xi)\,z\sqrt{(\xi^2-l^2)}\,K_1\{\sqrt{(\xi^2t^2-z^2l^2)}\}\,d\xi}{\sqrt{(\xi^2t^2-z^2l^2)}}, \tag{61}$$

$$= \frac{m}{\pi^2 V}\mathscr{R}\int_0^{\infty} e^{i\xi x}\frac{\xi\,(\operatorname{sgn}\xi)\,z\sqrt{(\xi^2-l^2)}\,K_1\{\sqrt{(\xi^2t^2-z^2l^2)}\}\,d\xi}{\sqrt{(\xi^2t^2-z^2l^2)}}. \tag{62}$$

This integral cannot be evaluated explicitly, and we have to proceed by deforming the contour.

When $x < 0$ we deform the path into the lower half-plane, as shown in figure 1, so that

$$\zeta = \int_0^{\infty} = \int_{I_1} + \int_{C_1}, \tag{63}$$

$$= \int_{I_1} = \zeta_N, \quad \text{say}, \tag{64}$$

since the circular part gives no contribution in the limit. Writing $\xi = -i\psi$, we have

$$\zeta_N = \frac{m}{2\pi V}\int_0^{\infty} e^{-\psi|x|}\frac{\psi z\sqrt{(\psi^2+l^2)}\,J_1\{\sqrt{(\psi^2t^2+z^2l^2)}\}\,d\psi}{\sqrt{(\psi^2t^2+z^2l^2)}}. \tag{65}$$

Here we have used the fact that

$$\mathscr{R}K_1(ir) = -(\tfrac{1}{2}\pi)\,J_1(r) \tag{66}$$

(from Watson 1944, § 3·7, equation (8)).

When $x > 0$, we deform the path similarly into the upper half-plane, and obtain

$$\zeta = \int_0^{\infty} = \int_{I_2} + \int_{C_2} + \int_{L_1, L_2}, \tag{67}$$

$$= \int_{I_2} + \int_{L_1, L_2}, \tag{68}$$

$$= \zeta_N + \zeta_W, \tag{69}$$

where for reasons to appear the subscript N stands for 'non-wave' and W for 'wave'. Putting $\xi = i\psi$ into the integral along I_2 gives the same value (65) for ζ_N.

For the case of neutral stability, for which $l = 0$, (65) is the complete solution, and can be integrated exactly (Watson 1944, § 13.2, equation (5)). This gives

$$\zeta = \frac{mz}{2\pi V(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{m}{2\pi V}\frac{\partial}{\partial z}\left\{\frac{-1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right\}. \tag{70}$$

This value is the same as the vertical velocity perturbation due to a source of strength m/V in a flow with uniform density and no gravitational forces, and provides a useful check on the present work.

When $l > 0$ (stable stratification), we will write (62) and (65) in a non-dimensional form by taking l^{-1} as unit of length and $V(0)$ as unit of velocity, so that $x^* = xl$, $\psi^* = \psi l^{-1}$, $V^* = V(z)/V(0)$, $m^* = ml^3/V(0)$, etc. Omitting the asterisks, the only visible difference in (62) and (65) is that l is replaced by unity.

If we now substitute into (65) from

$$\frac{J_1\{\sqrt{(\psi^2t^2+z^2)}\}}{\sqrt{(\psi^2t^2+z^2)}} = \sum_{\nu=0}^{\infty} \frac{(-\tfrac{1}{2}\psi^2t^2)^\nu J_{1+\nu}(z)}{\nu!\, z^{1+\nu}} \tag{71}$$

(Watson 1944, § 5.22, equation (1)), and expand the $\sqrt{(\psi^2+1)}$ in ascending powers of ψ, we can obtain an asymptotic expansion valid for large $|x|$:

$$\zeta_N \sim \frac{m}{2\pi V}\left[\frac{J_1(z)}{x^2} + \frac{3}{x^4}\left\{J_1(z) - t^2\frac{J_2(z)}{z}\right\} + \ldots\right]. \tag{72}$$

This expansion is adequate when y is not too large, but no suitable approximation has been found to cover the case when y is large.

We are more interested here in the wave term ζ_W, which comes from the integrals over the loops L_1, L_2, and which occurs only for $x > 0$. The integral here is difficult to deal with, particularly when y is too small or too large; when y is small the singularities at $\xi = z/t$ and $\xi = 1$ are close together, and when y is large the singularity at $\xi = z/t$ is near to that at the origin. We therefore consider first the case of medium values of y, for which

$$x\left(1-\frac{z}{t}\right) \gg 1, \quad \frac{xz}{t} \gg 1, \tag{73}$$

so that the singularities are all well separated. We can then write

$$\zeta_W = \int_{L_1} + \int_{L_2} = \int_{i\infty}^{(z/t+)} + \int_{i\infty}^{(1+)}, \tag{74}$$

and the dashed parts of L_1 and L_2 in figure 1 are included. Since we have assumed large x, the main contributions to the integrals over L_1 and L_2 come from small values of $\xi - z/t$ and $\xi - 1$, respectively, and we can neglect certain terms on this account. Consider first the integral over L_1. We substitute

$$\xi = \frac{z}{t} + \frac{\psi^2}{2tz}, \tag{75}$$

in (62), so that, from (73), we can neglect $\psi^2/2tz$ compared with $1 - z/t$ and z/t, and we have

$$\int_{L_1} \sim \frac{m}{\pi^2 V}\mathscr{R}\int_{L_3} \exp(izx/t + ix\psi^2/2tz)\left(\frac{z}{t}\right) z\left(\frac{z}{t}+1\right)^{\frac{1}{2}}\left(\frac{z}{t}-1\right)^{\frac{1}{2}} K_1(\psi)\frac{d\psi}{tz}, \tag{76}$$

$$= \frac{mz\,|y|}{\pi^2 V t^3}\mathscr{R}(-i)\, e^{izx/t}\int_{L_3} [e^{iX\psi^2} K_1(\psi)]\, d\psi, \tag{77}$$

where

$$X = \frac{x}{2tz} \tag{78}$$

and the contour L_3 is shown in figure 2. Now, since the integrand has no singularities other than at the origin,

$$\int_{L_3} [\ ]\, d\psi = \int_{-\infty}^{\infty} e^{iX\psi^2} K_1(\psi)\, d\psi, \tag{79}$$

$$= \int_0^{\infty} d\tau \int_{-\infty}^{\infty} \exp(iX\psi^2 - \psi\cosh\tau)\cosh\tau\, d\psi \tag{80}$$

according to Watson (1944, § 6.22, equation (5)). Putting

$$\psi = \frac{\cosh\tau}{2iX} + \nu, \tag{81}$$

and working out the ν-integral,

$$\int_{L_3} [\ ]\, d\psi = \sqrt{\left(\frac{\pi i}{X}\right)} \int_0^{\infty} \exp(-\cosh^2\tau/4iX)\cosh\tau\, d\tau, \tag{82}$$

$$= \frac{1}{2}\sqrt{\left(\frac{\pi i}{X}\right)}\, e^{-1/8iX} K_{\frac{1}{2}}\left(\frac{1}{8iX}\right), \tag{83}$$

again according to Watson (1944, § 6.22, equation (5)), first writing $\cosh^2\tau$ in terms of $\cosh 2\tau$. The Bessel function $K_{\frac{1}{2}}(1/8iX)$ can be expressed as a simple function by means of Watson (1944, § 3.71, equation (13)).

For the integral over L_2, we put

$$\xi = 1 + i\psi \tag{84}$$

in (62), and neglecting ψ terms in the same way as before we have

$$\int_{L_2} \sim \frac{mz}{\pi^2 V} \mathscr{R}\, e^{ix} \int_0^{\infty} e^{-\psi x} \frac{(2\psi)^{\frac{1}{2}} K_1(|y|)}{|y|} \{e^{3\pi i/4} - e^{-9\pi i/4}\}\, d\psi, \tag{85}$$

the curved part of L_2 giving no contribution. Thus for $x(1 - z/t) \gg 1$ and $xz/t \gg 1$, (74), (77), (83) and (85) give

$$\zeta_W \sim \frac{mz\,|y|}{\pi V t^3} \cos\left(\frac{xz}{t} + \frac{tz}{2x}\right) + \frac{2^{\frac{1}{2}} mz K_1(|y|)}{\pi^{\frac{3}{2}} V |y|\, x^{\frac{3}{2}}} \cos(x + 3\pi/4). \tag{86}$$

This result shows that the amplitude of the waves falls off fairly rapidly as y increases, so there is little point in trying to find a solution for still larger y. However, the solution for small y, and in particular for $y = 0$, is interesting, although mathematically difficult. For this case we assume simply $xz/t \gg 1$. Since the singularities at $\xi = z/t$ and $\xi = 1$ are now close together we have to take a single loop which passes round both, i.e. miss out the dashed parts of loops L_1, L_2 in figure 1. Thus, again making the substitution (75) we have

$$\zeta_W \sim \frac{mz(t+z)^{\frac{1}{2}}}{\pi^2 V t^{\frac{5}{2}}} \mathscr{R}\, e^{izx/t} \int_{L_4} e^{ix\psi^2/2tz} K_1(\psi) \left\{\frac{\psi^2}{2tz} - \left(1 - \frac{z}{t}\right)\right\}^{\frac{1}{2}} d\psi, \tag{87}$$

$$= \frac{mz^{\frac{1}{2}}(t+z)^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^2 V t^3} \mathscr{R}\, e^{izx/t} \int_{L_4} e^{iX\psi^2} K_1(\psi)\, \psi \left\{1 - \frac{Y^2}{\psi^2}\right\}^{\frac{1}{2}} d\psi, \tag{88}$$

where

$$Y = \sqrt{\{2z(t-z)\}} \tag{89}$$

and L_4 is shown in figure 2. Note that we can no longer neglect $\psi^2/2tz$ in comparison with $1-z/t$. We have made L_4 lie entirely outside the circle $|\psi| = Y$ because the method is now to expand $\surd(1-Y^2/\psi^2)$ as a convergent power series and integrate term by term. If Y is not too large (i.e. fairly small y), only two or

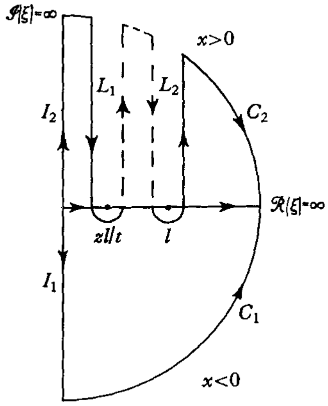

FIGURE 1. Contours in the ξ-plane.

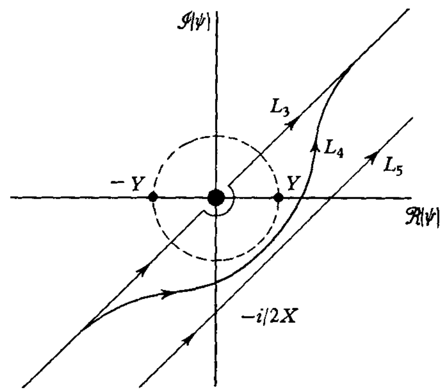

FIGURE 2. Contours in the ψ-plane.

three terms of this series will be needed to give a reasonably accurate result. However, the evaluation of the integrals is complicated, and is therefore given as an appendix to the paper. The result is

$$\zeta_W \sim \frac{mz^{\frac{1}{2}}(t+z)^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^2 V t^3}\,\mathscr{R}\, e^{ixz/t}\left\{f_0(X) - \frac{Y^2}{2}f_1(X) - \frac{Y^4}{8}f_2(X) - \ldots\right\}, \tag{90}$$

where X and Y are given by (78) and (89),

$$f_0(X) = \frac{(-i\pi)^{\frac{1}{2}}}{8X^{\frac{3}{2}}}e^{-1/8iX}\left\{K_0\left(\frac{1}{8iX}\right) + K_1\left(\frac{1}{8iX}\right)\right\}, \tag{91}$$

$$f_1(X) = \frac{(i\pi)^{\frac{1}{2}}}{4X^{\frac{1}{2}}}e^{-1/8iX}\left\{K_1\left(\frac{1}{8iX}\right) - K_0\left(\frac{1}{8iX}\right)\right\}, \tag{92}$$

and

$$f_2(X) = \frac{(i\pi)^{\frac{1}{2}}}{6}e^{-1/8iX}\left[K_1\left(\frac{1}{8iX}\right)\left\{iX^{\frac{1}{2}} + \frac{1}{2X^{\frac{1}{2}}}\right\} - K_0\left(\frac{1}{8iX}\right)\left\{3iX^{\frac{1}{2}} + \frac{1}{2X^{\frac{1}{2}}}\right\}\right]. \tag{93}$$

Any further terms can be worked out if required.

An alternative approximation can be found by the method of steepest descents. To do this we must deform the contour L_4 (in (88)) so that it lies entirely in the region where

$$K_1(\psi) \sim e^{-\psi}\sqrt{\left(\frac{\pi}{2\psi}\right)}. \tag{94}$$

The integral is then dominated by

$$\exp(iX\psi^2 - \psi) = e^{f(\psi)}. \tag{95}$$

The method of steepest descents is based on the fact that if some parameter, in this case x, is large, the main contribution to the integral is near the point where $f'(\psi) = 0$, i.e. the point $\psi = -i/2X$. The contour is accordingly deformed to pass through this point, which is a 'col' on the surface $z = \mathscr{R}\{f(\psi)\}$, so that $\mathscr{I}\{f(\psi)\}$ is constant all along it, and (to make sure the integral converges) so that it descends on each side of the col. $\mathscr{R}\{f(\psi)\}$ changes as rapidly as possible along this curve because it is perpendicular to the contours $\mathscr{R}\{f(\psi)\} = \text{const.}$, and the integrand does not oscillate rapidly along it, so we can approximate to the integral by considering the integrand near the col. The line of steepest descents in this case is shown as L_5 in figure 2.

For (94) to be true all along this path we need to have $1/2^{\frac{3}{2}}X$ large (i.e. $tz/2^{\frac{1}{2}}x$ large), so the approximation will only be good for large z, since x is already assumed large. Under these conditions we will usually have $Y < 1/2X$ ($\{2z(t-z)\}^{\frac{1}{2}} < tz/x$), and the line of steepest descents will pass outside the singularity at $\psi = Y$. If this is not so, the path would have to be deformed, but the descent would probably be steep enough if it were made to approach $+\infty$ along the real axis, the negative part being unchanged. The leading term of this approximation gives

$$\zeta_W \sim \frac{mz^{\frac{1}{2}}(t+z)^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi V t^3}\left\{\frac{t^2z^2}{x^2}+2z(t-z)\right\}^{\frac{1}{2}}\cos\left(\frac{xz}{t}+\frac{tz}{2x}\right). \tag{96}$$

In the case $y = 0$, (96) can be obtained directly from (90) and (91) by allowing $1/X$ to become large. Thus the approximations are consistent. It should be pointed out, however, that the value of (96), or any other approximation to (88) for large z, is limited, because in obtaining (88) terms have been omitted which, though small compared with those retained, may still be $O(1)$, and hence $O(X\psi^2)$, when z is large. From (96) it would appear that the amplitude of the waves tends to a finite value as z is increased, and this value is zero if V increases with height. This result would still be essentially true if we re-introduced the factor $\exp\{\frac{1}{2}(g/c^2+\beta)z\}$ (omitted after (38)), since its effect is small, although the amplitude might then theoretically increase indefinitely.

Each of the approximations shows waves with phase

$$\frac{xz}{t}+\frac{tz}{2x}. \tag{97}$$

In the region where the solutions hold the first term of (97) is dominant, and taking it alone gives lines of constant phase, and therefore wave crests and troughs, which in planes $z = \text{const.}$ are hyperbolic,

$$\frac{x^2}{C^2}-\frac{y^2}{z^2} = 1. \tag{98}$$

The phase C is the distance of the line from the origin along the x-axis. From (98), the waves are perpendicular to the stream on $y = 0$, but curve back downstream as y is increased. The asymptotes of all these hyperbolae pass through the origin and the angle they make with the x-axis, $\tan^{-1}(z/C)$, is less the greater the horizontal distance of the curve from the doublet, but greater at greater height. Hence, as we go downstream at a fixed height each wave curves back more rapidly than its predecessor, making the wavelength increase as y increases, but each wave curves back less rapidly as we go upwards at a fixed distance downstream.

The behaviour of the wave amplitude as y increases is peculiar. For $y = 0$ the waves decay downstream as shown by (90) and (91), like x^{-1} for small $x/2z^2$ (also given by (96)) and like $x^{-\frac{1}{2}}$ for large $x/2z^2$. As y is increased, the rate of decay is reduced, and when y is large enough for the first approximation (86) to hold, the waves, being mostly due to the first term, have more or less constant amplitude. The amplitude falls off fairly rapidly in the y-direction, like $y/(y^2+z^2)^{\frac{3}{2}}$. There seems to be no possible physical explanation for the changing rate of decay as y increases, or for the existence of the non-decaying waves, and it can only be assumed that the solution is inadequate because the doublet, being a point disturbance, has no significant length. Support for this reason can be shown by extending the solution, first to a line of doublets in the y-direction on the ground, and secondly, to a circular hill (§ 5).

The solution for the line of doublets is obtained by replacing y in (86) by $(y-y_1)$ and integrating with respect to y_1 from $-\alpha$ to α, for a line of length 2α. This gives

$$\zeta_W \sim \frac{m}{\pi x}\left[\sin\left(\frac{xz}{\{(y-\alpha)^2+z^2\}^{\frac{1}{2}}}\right)-\sin\left(\frac{xz}{\{(y+\alpha)^2+z^2\}^{\frac{1}{2}}}\right)\right] + \frac{2^{\frac{1}{2}}mz}{\pi^{\frac{3}{2}}Vx^{\frac{3}{2}}}\cos\left(x+\frac{3\pi}{4}\right)\left\{\int_{-\alpha}^{\alpha}\frac{K_1(|y-y_1|)}{|y-y_1|}\,dy_1\right\}, \tag{99}$$

and the solution is valid for $x(1-z/\sqrt{\{(y-\alpha)^2+z^2\}}) \gg 1$, $xz/\sqrt{\{(y+\alpha)^2+z^2\}} \gg 1$ and $2x^2/\{(y+\alpha)^2+z^2\} \gg 1$; the last condition arises because to perform the integration the $\cos\,(xz/t+tz/2x)$ in (86) has been replaced by $\cos\,(xz/t)$. These conditions imply that x must be large and that y must be large, but not too large, compared with α. We can see that the waves given by (99) decay like x^{-1}, and thus some of the peculiarities of the single doublet solution have been removed by introducing the significant length α into the disturbance. The actual size of α is not important, and it could be very small, so that the disturbance would be almost at a point. The other approximations cannot be integrated simply in this way, so the behaviour of the rate of decay with increase of y for the line of doublets cannot be shown.

4. The solution for an infinite ridge

Before going on to deal with the flow over a circular mountain it is convenient, for purposes of comparison, to write down a simple approximation to Queney's solution for the two-dimensional ridge (equation (1)) when l^2 is constant throughout the atmosphere. The Fourier Transform of (1) is

$$\zeta_0(0,\xi,\eta) = \tfrac{1}{2}Hb\exp\left(-b\,\xi\,\mathrm{sgn}\,\xi\right)\delta(\eta), \tag{100}$$

and to obtain the flow over the ridge we have to replace the $m/4\pi^2$ of (38) by the right-hand side of (100). The η-integral now gives unity, and so we have

$$\zeta = \frac{Hb}{2V}\int_{-\infty}^{\infty}\exp\{i\xi x - b\xi\,\mathrm{sgn}\,\xi - z\sqrt{(\xi^2-1)}\}\,d\xi, \tag{101}$$

$$= \frac{Hb}{V}\mathscr{R}\int_{0}^{\infty}\exp\{i\xi x - b\xi - z\sqrt{(\xi^2-1)}\}\,d\xi, \tag{102}$$

again using the non-dimensional co-ordinates. We deform the contour just as in the three-dimensional case, so that we have the wave term arising for $x > 0$, given by the singularity at $\xi = 1$. The non-wavy disturbance, ζ_N, is given by the integrals along the imaginary axis, and writing $\xi = (\operatorname{sgn} x)\, i\psi$, and approximating for large $|x|$ ($\psi \ll 1$), we get

$$\zeta_N \sim \frac{Hb}{V}\mathscr{R}\int_0^\infty \exp\{-\psi|x| - ib\,(\operatorname{sgn} x)\,\psi + iz\}\, i(\operatorname{sgn} x)\, d\psi, \tag{103}$$

$$= \frac{Hb}{V}\left(\frac{b\cos z - x\sin z}{x^2+b^2}\right). \tag{104}$$

This term has been obtained by Scorer (1953), but with his upper boundary condition, which leaves the singularity at $\xi = 1$ below the real axis, he gets a positive sign before the $x\sin z$. In this paper Scorer neglects the wave term on the grounds that the ridge is wide enough to make its amplitude small, but here we shall investigate it further.

The wave term is a loop integral, with $\int_{i\infty}^{(1+)}$ instead of $\int_0^\infty$ in (102), and putting $\xi = 1 + i\psi^2$ and again approximating for $x \gg 1$ ($\psi^2 \ll 1$), this gives

$$\zeta_W \sim \frac{2Hb}{V}\mathscr{R}\int_0^\infty \exp\{ix - b - \psi^2(x+ib)\}$$
$$\times\{-\exp(-z\sqrt{2\psi}\,e^{-3\pi i/4}) + \exp(-z\sqrt{2\psi}\,e^{\pi i/4})\}\,\psi i\, d\psi, \tag{105}$$

the contributions coming from the straight parts of the loop only. This can be written

$$\zeta_W \sim -\frac{4Hb}{V}\mathscr{R}\, i\, e^{ix-b}\int_0^\infty e^{-\psi^2(x+ib)}\sinh(z2^{\frac{1}{2}}e^{\frac{1}{4}\pi i}\psi)\,\psi\, d\psi, \tag{106}$$

$$= \frac{Hb}{V}\sqrt{(2\pi)}\, z\exp[-b + \{z^2 b/2(x^2+b^2)\}]$$
$$\times\mathscr{R}\left\{\frac{\exp(i[x + \{z^2x/2(x^2+b^2)\} - \frac{1}{4}\pi])}{(x+ib)^{\frac{3}{2}}}\right\}. \tag{107}$$

Lyra's solution (1943) for the small hill of rectangular cross-section is essentially the same as this solution with $b = 0$, $Hb = \text{const.}$ ($= m/\pi$), or, in other words, an infinite line doublet, because substituting these values throughout the working does not alter it in any other way. This gives the wave term

$$\zeta_W \sim \frac{2^{\frac{1}{2}}mz}{\pi^{\frac{1}{2}}Vx^{\frac{3}{2}}}\cos\left(x + \frac{z^2}{2x} - \frac{\pi}{4}\right), \tag{108}$$

which is equivalent to the approximation found by Queney (1947, equation (97)), but he has made a slight mistake in proceeding from his previous line. Queney points out that the approximations used in obtaining (105) are not very good unless $x^2 \gg z$, i.e. unless the height is not too great.

If x is large enough compared with b, the dependence of wave amplitude (in (107)) on the mountain size is essentially given by Hbe^{-b}. This factor is exactly the same as that given by the waves of the two-layer solution of Scorer (1949) (when his wave-number is taken as unity), and its behaviour has been discussed by Corby & Wallington (1956). We will return to this point later, when discussing results, in § 5. The line doublet wave term (108) gives greater amplitudes than the ridge of corresponding 'strength' (i.e. volume per unit depth, πHb equal to m), but the character of the waves appears to be the same. We shall now go on to show that in three dimensions this last statement is not true; the single-point doublet does not give waves of the same character as those given by a circular mountain.

5. The solution for a circular mountain

The extension of the solution to an isolated mountain, instead of a doublet, is difficult, because, for the solution to be any use, the η-integral of the inverse Fourier Transform has to be worked out exactly. It is for this reason that it has so far proved impossible to find a suitable elliptical or oval mountain, which would have given two parameters to vary, and thus this section has had to be restricted to a circular mountain. The mountain shape we shall use is given by equation (4), for which the Fourier transform is

$$\zeta_0(0,\xi,\eta) = \frac{1}{2\pi} Ha^2 \exp\{-a\sqrt{(\xi^2+\eta^2)}\}. \tag{109}$$

Here a is the radius of the hill at height $H/2^{\frac{3}{2}}$, whereas for the ridge (1), b is the half-width at height $\frac{1}{2}H$.

With this ground shape we get the double Fourier Integral

$$\zeta = \frac{Ha^2V(0)}{2\pi V(z)} \int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} \exp\left[i\eta y - z\{(\xi^2-l^2)(1+\eta^2/\xi^2)\}^{\frac{1}{2}} - a(\xi^2+\eta^2)^{\frac{1}{2}}\right] d\eta\, d\xi. \tag{110}$$

It is obvious that when $l = 0$ the flow is simply that due to a doublet of strength $m = 2\pi Ha^2V(0)$ at $z = -a$, for which

$$\zeta = \frac{Ha^2V(0)(z+a)}{V(z)\{x^2+y^2+(z+a)^2\}^{\frac{3}{2}}}. \tag{111}$$

When l is not zero but is still constant throughout the atmosphere, we make the same substitution as before (41), and the η-integral in (110) becomes, in the same non-dimensional units,

$$-\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-1)}} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \exp\left[i\xi y \sinh\tau - \{z\sqrt{(\xi^2-1)} + a\xi \operatorname{sgn}\xi\}\cosh\tau\right] d\tau, \tag{112}$$

$$= -\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-1)}} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \exp\{-r_3 \cosh(\tau - i\psi_3)\}\, d\tau, \tag{113}$$

where

$$r_3^2 = t_1^2\xi^2 - z^2 + 2az\xi \operatorname{sgn}\xi \sqrt{(\xi^2-1)}, \tag{114}$$

$$t_1^2 = y^2+z^2+a^2, \tag{115}$$

and

$$\tan\psi_3 = \xi y / \{z\sqrt{(\xi^2-1)} + a\xi \operatorname{sgn}\xi\}. \tag{116}$$

This time r_3^2 is never real and negative (except at $\xi = 0$, which will be a singular point anyway), so we can define $\mathscr{R}(r_3) > 0$ everywhere. Moreover, with $a \neq 0$ we can see from (112) that the integrand $\to 0$ as $\tau \to \infty$ (for real ξ), so that (113) is

$$-\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-1)}} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \exp(-r_3 \cosh p_3)\, dp_3, \tag{117}$$

$$= -\frac{\xi \operatorname{sgn}\xi}{\sqrt{(\xi^2-1)}} \frac{\partial}{\partial z} \{2K_0(r_3)\}, \tag{118}$$

according to Watson (1944, § 6.22, equation (5)). We then have

$$\zeta = \frac{Ha^2}{\pi V}\int_{-\infty}^{\infty} e^{i\xi x}\,\xi\,\mathrm{sgn}\,\xi\,\{z\sqrt{(\xi^2-1)}+a\xi\,\mathrm{sgn}\,\xi\} \times \frac{K_1[\sqrt{\{\xi^2t_1^2-z^2+2az\xi\,\mathrm{sgn}\,\xi\sqrt{(\xi^2-1)}\}}]\,d\xi}{\sqrt{\{\xi^2t_1^2-z^2+2a\xi\,\mathrm{sgn}\,\xi\sqrt{(\xi^2-1)}\}}}. \tag{119}$$

This integral has a very important difference from the corresponding one of the doublet solution (61), namely, that the argument r_3 of the Bessel function is never zero ($a \neq 0$). This is equivalent to saying that

$$\xi^2t_1^2-z^2+2az\xi\sqrt{(\xi^2-1)} \tag{120}$$

is never zero when $\mathscr{R}(\xi) > 0$ and $\mathscr{R}\{\sqrt{(\xi^2-1)}\} > 0$. For (120) to be zero we must have

$$\xi^4t_1^4-2z^2t_1^2\xi^2+z^4 = 4a^2z^2\xi^2(\xi^2-1), \tag{121}$$

which gives

$$\xi^2 = \frac{z^2}{D}\{y^2+z^2-a^2\pm 2ia\,|y|\}, \tag{122}$$

where

$$D = (a^2-z^2)^2+2y^2(a^2+z^2)+y^4 > 0, \tag{123}$$

and hence

$$\xi^2t_1^2-z^2 = \frac{2z^2}{D}\{a^2(z^2-a^2)-a^2y^2\pm ia\,|y|\,t_1^2\}. \tag{124}$$

If we take the upper signs, (122) shows that $\mathscr{I}(\xi^2) > 0$, and since $\mathscr{R}(\xi)$, $\mathscr{R}\{\sqrt{(\xi^2-1)}\} > 0$, $\mathscr{I}(\xi) > 0$ and $\mathscr{I}\{\sqrt{(\xi^2-1)}\} > 0$, so that $\mathscr{I}\{\xi\sqrt{(\xi^2-1)}\} > 0$. But by (124) $\mathscr{I}(\xi^2t_1^2-z^2) > 0$ also, and therefore (120) is never zero because it has positive imaginary part. Similarly, if the lower signs are taken, (120) has negative imaginary part. These imaginary parts can only be zero if $y = 0$, in which case $\xi^2 = z^2/(z^2-a^2)$, and if $z > a$, (120) is obviously positive. If $y = 0$ and $z < a$, $\xi = \pm iz/\sqrt{(a^2-z^2)}$ and $\sqrt{(\xi^2-1)} = \pm ia/\sqrt{(a^2-z^2)}$, and whether upper or lower signs be taken, (120) is negative and non-zero. The fact that r_3 is never zero is a considerable simplification from the doublet integral, because it means that we now have only one pair of singularities, at $\xi = \pm 1$, which produce waves, and thus we have no difficulties arising from the behaviour due to singularities which are close together.

The method of dealing with (119) is essentially the same as before. The non-wavy component of the disturbance, ζ_N, is again given by integrals I_1 or I_2, along the imaginary axis of the ξ-plane, and

$$\zeta_N = \frac{Ha^2}{V}\int_0^{\infty}\exp(-\psi x\,\mathrm{sgn}\,x)\,\psi\{z\sqrt{(\psi^2+1)}-(\mathrm{sgn}\,x)\,a\psi\} \times \frac{J_1[\sqrt{\{\psi^2t_1^2+z^2-(\mathrm{sgn}\,x)\,2az\psi\sqrt{(\psi^2+1)}\}}]\,d\psi}{\sqrt{\{\psi^2t_1^2+z^2-(\mathrm{sgn}\,x)\,2az\psi\sqrt{(\psi^2+1)}\}}}. \tag{125}$$

It is interesting to note that the up-stream and down-stream components of this part of the disturbance are no longer symmetrical. An asymptotic expansion for (125), found in exactly the same way as for the doublet solution, is

$$\zeta_N \sim \frac{Ha^2}{V}\left[\frac{J_1(z)}{x^2}-\frac{2a}{x^3}\left\{\frac{J_1(z)}{z}-J_2(z)\right\}+\frac{3}{x^4}\left\{J_1(z)-\frac{J_2(z)}{z}(2a^2+t_1^2)+a^2J_3(z)\right\} + \frac{12a}{x^5}\left\{J_2(z)\left(2+\frac{t_1^2}{z^2}\right)-\frac{J_3(z)}{z}(a^2+t_1^2)+\frac{a^2}{3}J_4(z)\right\}\right]+\ldots. \tag{126}$$

This is valid for large x, but will not be adequate when y or a is large. However, as pointed out above, the present paper is more concerned with the wave terms of the solution, and Scorer (1956) has already worked out the non-wavy terms for the flow over circular and oval hills when $l^2 = \text{const}$. Scorer obtains his solution by his method of integrating up the flow over infinite ridges at all angles to the flow.

To find an approximation to the wave term, now given by a single loop integral (L_2), we put

$$\xi = 1+\frac{(y^2+a^2)}{2a^2z^2}\psi^2, \tag{127}$$

and approximate for large x, expanding the terms in ascending powers of ψ. All terms up to $O(\psi^3)$ have been retained. This gives

$$\zeta_W \sim \frac{2H(y^2+a^2)^{\frac{1}{2}}}{\pi V a z^2}\mathscr{R}\, e^{ix}\int_{L_3} e^{iX_1\psi^2}\psi\left\{a^2+(y^2+a^2)^{\frac{1}{2}}\psi+\frac{(y^2+a^2)}{z^2}\psi^2+\frac{5(y^2+a^2)^{\frac{3}{2}}}{8a^2z^2}\psi^3+\dots\right\}$$
$$\times\frac{K_1\left[(y^2+a^2)^{\frac{1}{2}}\left\{1+\dfrac{2\psi}{(y^2+a^2)^{\frac{1}{2}}}+\dfrac{t_1^2\psi^2}{a^2z^2}+\dfrac{5(y^2+a^2)^{\frac{1}{2}}}{4a^2z^2}\psi^3+\dots\right\}^{\frac{1}{2}}\right]d\psi}{\left\{1+\dfrac{2\psi}{(y^2+a^2)^{\frac{1}{2}}}+\dfrac{t_1^2\psi^2}{a^2z^2}+\dfrac{5(y^2+a^2)^{\frac{1}{2}}}{4a^2z^2}\psi^3+\dots\right\}^{\frac{1}{2}}}, \tag{128}$$

where

$$X_1 = \frac{x(y^2+a^2)}{2a^2z^2} \tag{129}$$

and L_3 is the same contour as before (figure 2). We can now expand the remaining square roots if

$$\frac{2\psi}{(y^2+a^2)^{\frac{1}{2}}}+\frac{t_1^2\psi^2}{a^2z^2}+\frac{5(y^2+a^2)^{\frac{1}{2}}}{4a^2z^2}\psi^3 \ll 1, \tag{130}$$

that is, if

$$\frac{2^{\frac{1}{2}}x^{\frac{3}{2}}(y^2+a^2)}{az(5+4x)+2^{\frac{3}{2}}t_1^2x^{\frac{1}{2}}} \gg 1. \tag{131}$$

This condition restricts the range of the solution considerably, especially when z is large, but we will still be able to see the more important characteristics of the solution. The behaviour for large z has not been found, and in any case the approximations involved in reducing the wave term to the form (128) are not very good when z is too large or when x^2 is not $\gg t_1^2$, since in these cases terms which are $O(1)$ have again been neglected. If (131) holds, then, (128) can be rewritten as

$$\zeta_W \sim \frac{2H(y^2+a^2)^{\frac{1}{2}}}{\pi V a z^2}\mathscr{R}\, e^{ix}\int_{L_3} e^{iX_1\psi^2}\psi\left[a^2+\frac{y^2\psi}{(y^2+a^2)^{\frac{1}{2}}}+\left\{\frac{y^2+a^2}{2z^2}-\frac{3y^2}{2(y^2+a^2)}\right\}\psi^2\right.$$
$$\left.+\left\{\frac{a^2y^2-4z^2y^2+y^4}{8a^2z^2(y^2+a^2)^{\frac{1}{2}}}+\frac{5y^2}{2(y^2+a^2)^{\frac{3}{2}}}\right\}\psi^3+\dots\right]K_1(\Lambda)\,d\psi, \tag{132}$$

where

$$\Lambda = (y^2+a^2)^{\frac{1}{2}}(1+\chi), \tag{133}$$

and

$$\chi = \frac{\psi}{(y^2+a^2)^{\frac{1}{2}}}+\left\{\frac{t_1^2}{2a^2z^2}-\frac{1}{2(y^2+a^2)}\right\}\psi^2+\left\{\frac{y^2+a^2-4z^2}{8a^2z^2(y^2+a^2)^{\frac{1}{2}}}+\frac{1}{2(y^2+a^2)^{\frac{3}{2}}}\right\}\psi^3+\dots \tag{134}$$

The next step is to expand the Bessel Function in a Taylor Series:

$$K_1(\Lambda) = K_1\{(y^2+a^2)^{\frac{1}{2}}\} + (y^2+a^2)^{\frac{1}{2}}\,\chi K_1'\{(y^2+a^2)^{\frac{1}{2}}\} + \dots \tag{135}$$

This series is convergent only for $|\chi| < 1$, but if we make

$$\{(2x)^{\frac{3}{2}}(y^2+a^2)^3\}/\{(y^2+a^2)^2[az(5+4x)+2^{\frac{3}{2}}t_1^2x^{\frac{1}{2}}] - (y^2+a^2)\,az[4t_1^2+2^{\frac{3}{2}}azx^{\frac{1}{2}}]+4a^3z^3\} \gg 1 \tag{136}$$

then $|\chi| \ll 1$ for all the ψ which contribute significantly to the integral, and the approximation will be valid. Fortunately, the condition (136) is satisfied by all combinations of x, y, z, a which satisfy (131), and is consequently no additional restriction on the range of the solution. This method of solution will not work, however, if both y and a are zero, since then the Bessel functions on the right-hand side of (135) become infinite. The case $a = 0$ is really that of the doublet solution, but putting $a = 0$, $2\pi Ha^2 = m$ at this stage will not give the doublet answers because the doublet integral has the extra singularity. The limit as $a \to 0$ of the asymptotic solution for the mountain is not the same as the asymptotic solution for the limit as $a \to 0$.

Substituting the Taylor series (135) into (132), only the even powers of ψ contribute to the integral, and

$$\begin{aligned}\zeta_W \sim\ & \frac{2^{\frac{3}{2}}Ha^2z\cos(x+3\pi/4)}{\pi^{\frac{1}{2}}V(y^2+a^2)\,x^{\frac{3}{2}}}\left[\left\{\frac{y^2-a^2}{(y^2+a^2)^{\frac{1}{2}}}\right\}K_1\{\surd(y^2+a^2)\} - a^2K_0\{\surd(y^2+a^2)\}\right] \\ & + \frac{2^{\frac{1}{2}}6Ha^4z^3\cos(x-3\pi/4)}{\pi^{\frac{1}{2}}V(y^2+a^2)^2x^{\frac{5}{2}}}\left[\left\{\frac{7y^2-a^2}{2(y^2+a^2)} - \frac{a^4+4y^4+4z^2y^2+5a^2y^2}{8a^2z^2} - \frac{a^2}{6}\right\}K_0\{\surd(y^2+a^2)\}\right. \\ & \left. + \left\{\frac{7y^2-a^2}{(y^2+a^2)^{\frac{3}{2}}} + \frac{3a^4-3y^4-8y^2z^2}{8a^2z^2(y^2+a^2)^{\frac{1}{2}}} + \frac{3a^4+3y^4+6y^2a^2+6y^2z^2-2a^2z^2}{6z^2(y^2+a^2)^{\frac{1}{2}}}\right\}K_1\{\surd(y^2+a^2)\}\right] \\ & + O(x^{-\frac{7}{2}}).\end{aligned} \tag{137}$$

Recurrence formulae (Watson 1944, § 3.71, equation (3)) have been used to write the derivatives of the Bessel function in terms of K_0 and K_1. Both the terms of this series are exact, and terms of orders higher than ψ^3, which have been omitted from (128), contribute only to terms of at most $O(x^{-\frac{7}{2}})$ in this solution.

The waves shown by (137) ultimately decay like $x^{-\frac{3}{2}}$, as do those of the infinite ridge solution (107). The rate of decay no longer varies as y increases, and non-decaying waves do not appear. The mountain, by introducing the length a into the problem, has removed one of the singularities of the doublet integral and, with it, the peculiar behaviour. The first term of (137) is directly related to the second term of the doublet approximation for larger y (86), which comes from the same singularity. Figure 3 shows the waves in the central plane $yl = 0$ for a circular hill with $Hl = \frac{1}{2}$ and $al = 2$, and also the shape and position of the wave crests in the horizontal plane $zl = 1$, for the same hill. In figure 3 the main-stream velocity $V(z)$ has been taken to be constant, so that the amplitude of the waves increases with height as far as the solution holds. Including the additional factor $\exp\{\frac{1}{2}(g/c^2+\beta)z\}$ would make no significant difference. Figure 3 has been computed from both terms of (137), and shows these wave terms only. The non-wavy part of the disturbance, ζ_N (126), has been left out so that the waves can be seen more clearly. Figures 4, 5 and 6, below, have been computed using only the first term of (137). Including the second term has little effect on the actual amplitude, but alters the positions of the waves relative to the mountain; in fact, the co-ordinate y in figures 4, 5 and 6 should really be taken as a curvilinear co-ordinate parallel to the wave crests, but the effect of this is also small, as can be seen from figure 3.

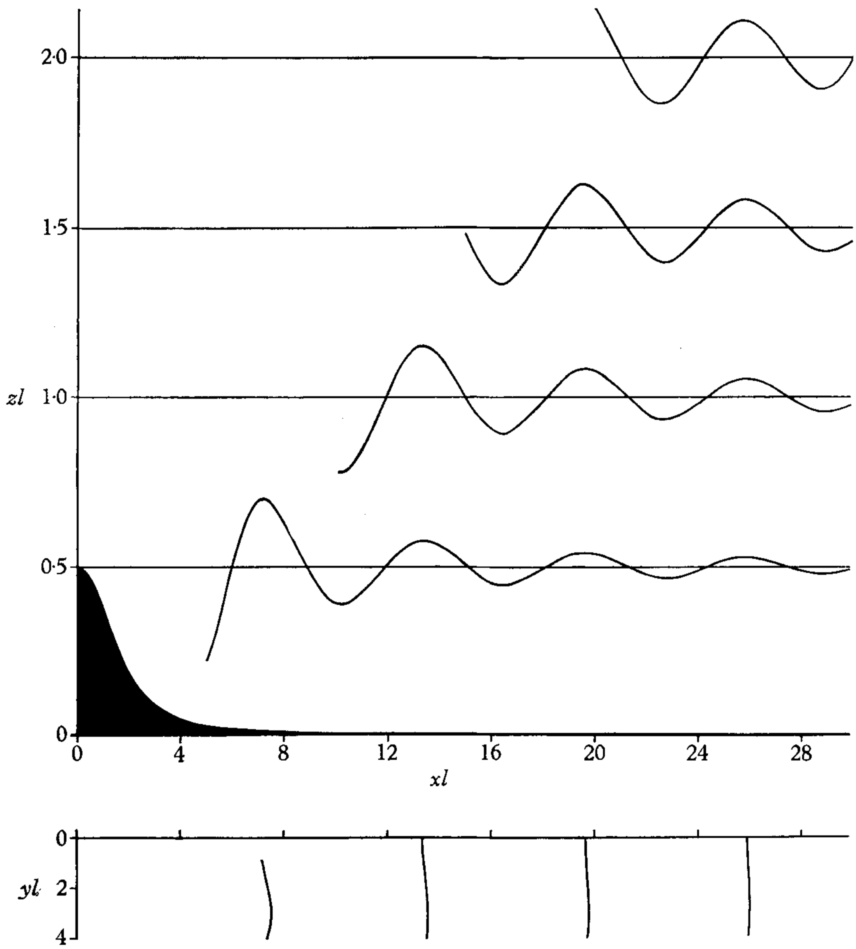

FIGURE 3. Waves in the central plane $yl = 0$, as shown by the theory for a circular mountain with $Hl = \frac{1}{2}$ and $al = 2$, when main-stream velocity $V(z)$ and static stability parameter $\beta(z)$ are both constant. The vertical scale for the waves is $12\frac{1}{2}$ times the vertical scale for the mountain and for the undisturbed streamlines. The lower diagram shows the shape of the wave crests in the plane $zl = 1$, for the same mountain. The approximations do not hold right up to $yl = 0$ on the first crest.

In the y-direction the amplitude of the waves given by the circular mountain falls off very rapidly, because of the Bessel functions, and this is shown for various values of al in figure 4. The rate of fall-off appears to depend primarily on a and is independent of x, so that the waves are not contained in a wedge, like ship waves, but in a strip. The width of the strip is determined by a and is roughly equal to $2a$.

Perhaps the most interesting feature of (137) is the dependence of amplitude on mountain width. Figures 5 and 6 show the amplitude as al is varied with height constant, and as al and Hl are varied in proportion, preserving the mountain shape, respectively, for various planes y = const. These figures also show the

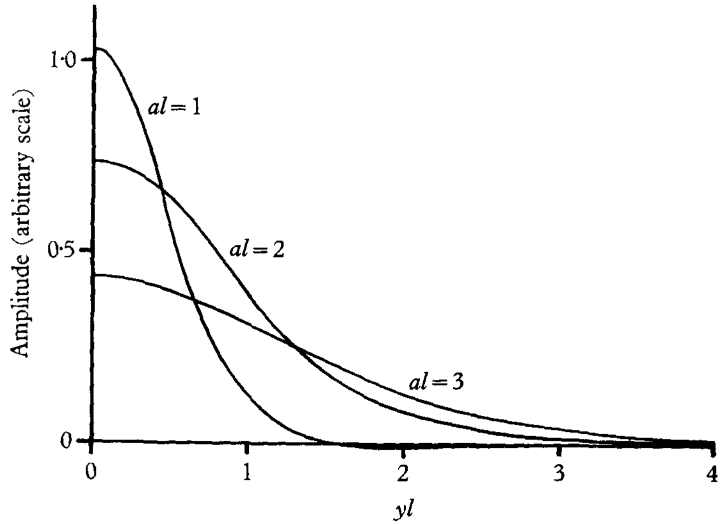

FIGURE 4. Fall-off of amplitude of waves as yl increases, for circular mountains with various values of al.

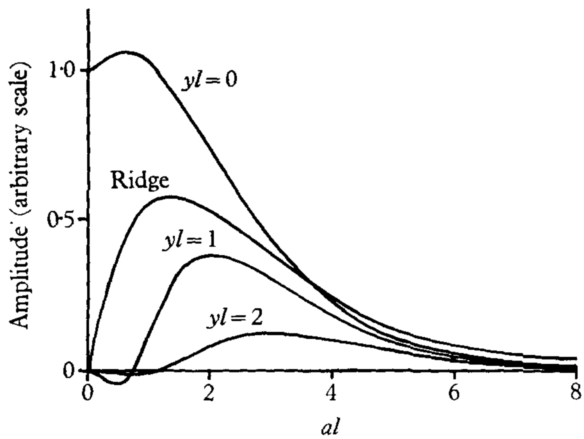

FIGURE 5. Dependence of wave amplitude on mountain width al, when mountain height is kept constant, for various planes yl = const., and for the infinite ridge with the same values of al.

corresponding curves (on the same scale of amplitude) for the infinite ridge with the same values of al. (The value of bl, the half width at height $\frac{1}{2}Hl$, of the mountain is only slightly greater than that of the ridge with the same al.) Each of these curves has a fairly sharp peak, so the idea, due to Corby & Wallington (1956), that a 'resonance' of the air stream with the mountain is needed to produce large amplitude waves applies equally well in two or in three dimensions, and in one- or two-layer solutions. As yl increases, the value of al which gives maximum amplitude (for the circular mountain waves) increases and the peaks become less sharp, but as the amplitude falls off very rapidly this is unimportant and only the $yl = 0$ values will be relevant. Waves of maximum amplitude (always in the plane $y = 0$) are given by $al = 0{\cdot}6$ for a mountain of fixed height (figure 5) and by $al = 2{\cdot}1$ for a mountain of fixed shape (figure 6). The corresponding figures for the ridges, $al = 1{\cdot}35$ and $al = 2{\cdot}70$ ($bl = 1$ and $bl = 2$) show that they are wider than the optimum circular mountains, and the peaks of the curves given by the ridge are not so sharp as those given by the mountain in plane $yl = 0$.

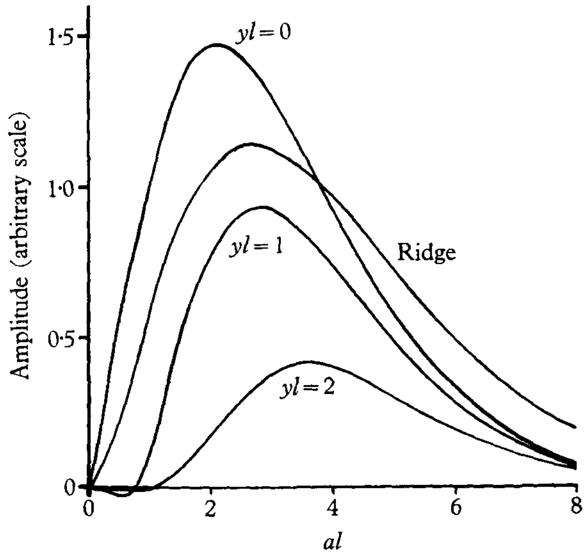

FIGURE 6. Dependence of wave amplitude on mountain width al, when height Hl and width are varied in the same proportion, for various planes yl = const., and for the infinite ridge with the same values of al.

These curves also show that for any al less than about 3·75, the waves produced by the mountain in the plane $yl = 0$ have greater amplitude than those due to the ridge, although the mountain waves in planes $yl = 1$ and $yl = 2$ are always smaller. The fact that the circular mountain can give rise to bigger waves than the ridge is probably because the waves do not spread out in a wedge, and thus all the energy of the waves is concentrated in a narrow strip immediately behind the mountain. As al becomes larger, the concentration of the energy is reduced (cf. figure 4) and eventually all the waves have to have amplitude smaller than those produced by the ridge.

The amplitudes all tend to zero as al becomes large, the waves due to the circular hill falling off slightly more rapidly than those due to the infinite ridge. As pointed out at the end of § 4, the curves for the infinite ridge solution in figures 5 and 6 are exactly the same as those for Scorer's two layer solution (1949), so that neglecting waves on the grounds that al (or bl) is reasonably large can only be justified if the waves of each type of solution are neglected. It would appear that any mountain will produce some form of lee waves in any stably stratified air stream. Whether the waves can be neglected or not depends entirely on the width of the mountain or ridge in relation to the wavelength, for all types of waves.

Appendix

For small y, when the wave-producing singularities are close together, we have, expanding the factor $\sqrt{(1-Y^2/\psi^2)}$ in (88),

$$\zeta_W \sim \frac{mz^{\frac{1}{2}}(t+z)^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^2 V t^3}\,\mathscr{R}\, e^{ixz/t}\left\{f_0(X)-\frac{Y^2}{2}f_1(X)-\frac{Y^4}{8}f_2(X)-\dots\right\}, \tag{90}$$

and we have to evaluate integrals of the form

$$f_n(X) = \int_{L_4} \psi^{1-2n}\, e^{iX\psi^2} K_1(\psi)\, d\psi, \tag{138}$$

for $n = 0, 1, 2, \dots$.

When $n = 0$ we can integrate by parts, and we have

$$f_0(X) = \int_{L_4} e^{iX\psi^2} K_0(\psi)\, d\psi + 2iX\int_{L_4} \psi^2 e^{iX\psi^2} K_0(\psi)\, d\psi, \tag{139}$$

$$= \left[1+2X\frac{d}{dX}\right]\int_{L_4} e^{iX\psi^2} K_0(\psi)\, d\psi. \tag{140}$$

The integral

$$F(X) = \int_{L_4} e^{iX\psi^2} K_0(\psi)\, d\psi \tag{141}$$

can be worked out in exactly the same way as (79), and is

$$F(X) = \frac{1}{2}\sqrt{\left(\frac{i\pi}{X}\right)}\, e^{-1/8iX}\, K_0\!\left(\frac{1}{8iX}\right). \tag{142}$$

With (140), this gives $f_0(X)$ as shown in (91). This is the only term needed when $y = 0$.

The next integral, $f_1(X)$, can be found quite easily by integrating $f_0(X)$ by parts again, in a different way

$$f_0(X) = \frac{1}{2iX}\int_{-\infty}^{\infty} \frac{e^{iX\psi^2} K_1(\psi)\, d\psi}{\psi} + \frac{1}{2iX}\int_{-\infty}^{\infty} e^{iX\psi^2} K_0(\psi)\, d\psi, \tag{143}$$

$$= \frac{1}{2iX}\{f_1(X)+F(X)\}. \tag{144}$$

As $f_0(X)$ and $F(X)$ are known this gives $f_1(X)$ (92) immediately.

The remaining $f_n(X)$ can now be deduced from (144) and the fact that, for any n,

$$f_{n+1}(X) = i\int f_n(X)\, dX. \tag{145}$$

For instance, integrating (144) once,

$$i\int 2iXf_0(X)\, dX = f_2(X) + i\int F(X)\, dX. \tag{146}$$

Integration by parts gives

$$i\int 2iXf_0(X)\, dX = i2iX\int f_0(X)\, dX - i\int 2i\int f_0(X)\, dX\, dX, \tag{147}$$

$$= 2iXf_1(X) - 2f_2(X), \tag{148}$$

and, from (140), we have

$$f_1(X) = i\int\left[1+2X\frac{d}{dX}\right]F(X)\,dX, \tag{149}$$

$$= 2iXF(X)-i\int F(X)\,dX. \tag{150}$$

Combining (146), (148) and (150), we get

$$f_2(X) = \tfrac{1}{3}\{(2iX+1)f_1(X)-2iXF(X)\}, \tag{151}$$

which gives (93).

The author wishes to thank Professor M. J. Lighthill for his guidance and inspiration throughout the preparation of the paper. The author is indebted to the Department of Scientific and Industrial Research for a maintenance grant.

REFERENCES

CORBY, G. A. 1954 *Quart. J. Roy. Met. Soc.* **80**, 491.
CORBY, G. A. & SAWYER, J. S. 1958*a* *Quart. J. Roy. Met. Soc.* **84**, 25.
CORBY, G. A. & SAWYER, J. S. 1958*b* *Quart. J. Roy. Met. Soc.* **84**, 284.
CORBY, G. A. & WALLINGTON, C. E. 1956 *Quart. J. Roy. Met. Soc.* **82**, 266.
LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.
LIGHTHILL, M. J. 1957 *J. Fluid Mech.* **3**, 113.
LYRA, G. 1943 *Z. angew. Math. Mech.* **23**, 1.
QUENEY, P. 1947 *Misc. Rep., Dept. Met., University of Chicago*, no. 23.
QUENEY, P. 1948 *Bull. Amer. Met. Soc.* **29**, 16.
SCORER, R. S. 1949 *Quart. J. Roy. Met. Soc.* **75**, 41.
SCORER, R. S. 1953 *Quart. J. Roy. Met. Soc.* **79**, 70.
SCORER, R. S. 1954 *Quart. J. Roy. Met. Soc.* **80**, 417.
SCORER, R. S. 1955 *Quart. J. Roy. Met. Soc.* **81**, 340.
SCORER, R. S. 1956 *Quart. J. Roy. Met. Soc.* **82**, 75.
SCORER, R. S. 1958 *Quart. J. Roy. Met. Soc.* **84**, 182.
SCORER, R. S. & WILKINSON, M. 1956 *Quart. J. Roy. Met. Soc.* **82**, 419.
WATSON, G. N. 1944 *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press.